Syntax & Semantics WS2019/2020
Lecture 21: Predicate Logic

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Semantics Lectures

- Lecture 18: Introduction to Semantics
  Kroeger (2019). Chapters 1-2.
- Lecture 19: Word Meaning
  Kroeger (2019). Chapter 5-6.
- Lecture 20: Propositional Logic
  Kroeger (2019). Chapter 3-4; and Zimmermann &
  Sternefeld Chapter 7.
- **Lecture 21: Predicate Logic**
  Kroeger (2019). Chapter 4; and Zimmermann &
  Sternefeld Chapter 10 (p. 244-258).

Section 1: Recap of Lecture 20

Formal Definition: Extensions

"Let us denote the **extension** of an expression A by putting double brackets '[[]]' around A, as is standard in semantics. The extension of an expression depends on the **situation s** talked about when uttering A ; so we add the index s to the closing bracket."

Zimmermann & Sternefeld (2013), p. 85.

$[[\text{Paul}]]_s$ = Paul McCartney[1]
$[[\text{the biggest German city}]]_s$ = Berlin
$[[\text{table}]]_s$ = {table_1, table_2, table_3, ..., table_n}[2]
$[[\text{sleep}]]_s$ = {sleeper_1, sleeper_2, sleeper_3, ..., sleeper_n}
$[[\text{eat}]]_s$ = {$\langle \text{eater}_1, \text{eaten}_1 \rangle$, $\langle \text{eater}_2, \text{eaten}_2 \rangle$, ..., $\langle \text{eater}_n, \text{eaten}_n \rangle$}

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[1]Zimmermann & Sternefeld just put the full proper name in brackets here, Kroeger follows another convention and just put the first letter in lower case, e.g. $[[\text{p}]]_s$.

[2]Kroeger (2019) uses upper case notation for both nouns and predicates, e.g. TABLE and SLEEP respectively.

Formal Definition: Proposition

"The **proposition expressed by a sentence** is the **set of possible cases [situations]** of which that sentence is true."

Zimmermann & Sternefeld (2013), p. 141.

Coin-flip example:

situation	flip1	flip2
1	heads	heads
2	tails	tails
3	heads	tails
4	tails	heads

Sentence	**Proposition**
S_1: only one flip landed heads up	$[\![S_1]\!] = \{3, 4\}$
S_2: all flips landed heads up	$[\![S_2]\!] = \{1\}$
S_3: flips landed at least once tails up	$[\![S_3]\!] = \{2, 3, 4\}$
etc.	etc.

Types of Inferences

There are (at least) **three types of inferences** that are relevant for analyzing sentence meanings:

- ► Inferences based on **content words**
- ► Inferences based on **logical words** (rather than content words)
- ► Inferences based on **quantifiers** (and logical words)

Kroeger (2019). Analyzing meaning, p. 56.

Logical Word Inference

If inferences are drawn based purely on the **meaning of logical words** (operators), then the inference is generalizable to a potentially infinite number of premisses and conclusions. Note that we can replace the propositions by placeholders. Here, we are in the domain of **propositional logic**.

(1) Premise 1: **Either** *Joe is crazy* **or** *he is lying.*
Premise 2: *Joe is* **not** *crazy.*

—————————————————————————————

Conclusion: **Therefore**, *Joe is lying*.

(2) Premise 1: **Either** *x* **or** *y.*
Premise 2: **not** *x.*

—————————————————————————————

Conclusion: **Therefore**, *y*.

Propositional Operators

We will here use the following operators:

Operator	Alternative Symbols	Name	English Translation
$\neg$	$\sim$, !	negation	*not*
$\wedge$	., &	conjunction	*and*
$\vee$	+, $\|$	disjunction (inclusive *or*)	*or*
XOR	EOR, EXOR, $\oplus$, $\underline{\vee}$	exclusive *or*	*either ... or*
$\rightarrow$	$\Rightarrow$, $\supset$	material implication[3]	*if ..., then*
$\leftrightarrow$	$\Leftrightarrow$, $\equiv$	material equivalence[4]	*if, and only if ..., then*

Note: We will here assume that the English translations and the operators themselves are indeed equivalent in their meanings. However, in language usage, this might not actually be the case.

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[3]aka *conditional*
[4]aka *biconditional*

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Truth Tables

In a **truth table** we identify the extensions of (declarative) sentences as truth values. In the notation typically used, the variables p and q represent such **truth values of sentences**.[5] The left table below gives the notation according to Zimmermann & Sternefeld, the right table according to Kroeger. We will use the latter for simplicity.

$[\![S_1]\!]_s$	$[\![S_2]\!]_s$	$[\![S_1]\!]_s \wedge [\![S_2]\!]_s$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

---

[5]Kroeger (2019), p. 58 writes that p and q are variables that represent propositions. However, according to the definitions we have given above this is strictly speaking not correct.

Building Truth Tables for Complex Sentences

We will follow the following four steps to analyze the sentence below:

1. Identify the **logical words** and translate them into **logical operators**

2. **Decompose the sentence** into its component declarative parts and assign **variables** to them (i.e. p and q).

3. Translate the whole sentence into **propositional logic notation**

4. Start the truth table with the variables (i.e. p and q) **to the left**, and then add operators step by step (from the most embedded to the outer layers).

Example Sentence: *If the president is either crazy or he is lying, and it turns out he is lying, then he is not crazy.*

Beyond Propositional Logic

"The propositional logic outlined in this section is an important part of the logical metalanguage for semantic analysis, but it is not sufficient on its own because it is concerned only with **truth values** [of whole sentences]. We need a way to go beyond p and q, to represent the actual meanings of **the basic propositions** we are dealing with."

Kroeger (2019). Analyzing meaning, p. 66.

Example Sentences (Set 1):

p: John is hungry.
q: John is smart.
r: John is my brother.

Example Sentences (Set 2):

p: John snores.
q: Mary sees John.
r: Mary gives George a cake.

Note: Propositional logic assigns variables (p, q, r) to whole declarative sentences, and hence is "blind" to the fact that the first set of sentences shares both the same subject, and the copula construction, whereas the second set of sentences uses predicates of different valencies and different subjects and objects.

Beyond Propositional Logic

A second major limitation of propositional logic is that it cannot take into account **quantifications**, and hence cannot decide on the truth values of the classical syllogisms below.

(3) Premise 1: ***All** men are mortal.*
     Premise 2: *Socrates is a man.*
     _____

     Conclusion: *Therefore, Socrates is mortal*.

(4) Premise 1: *Arthur is a lawyer.*
     Premise 2: *Arthur is honest.*
     _____

     Conclusion: *Therefore, **some (= at least one)** lawyer is honest*.

Section 2: Historical Notes

Historical Perspective

"The first formulation of predicate logic can be found in Frege (1879); a similar system was developed independently by Peirce (1885). Modern version radically differ from these ancestors in notation but not in their expressive means."

Zimmermann & Sternefeld (2013), p. 244.

Predicate Logic

| 1860 | 1870 | 1880 | 1890 | 1900 | 1910 | 1920 |

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"[...] fand ich ein Hindernis in der **Unzulänglichkeit der Sprache**, die bei aller entstehenden Schwerfälligkeit des Ausdruckes doch, je verwickelter die Beziehungen wurden, desto weniger die Genauigkeit erreichen liess, welche mein Zweck verlangte. Aus diesem Bedürfnisse ging der Gedanke der vorliegenden **Begriffsschrift** hervor."

Frege (1879). Begriffsschrift: Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens, p. X.

Translation: [...] I found the **inadequacy of language** to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This **deficiency** led me to the idea of the present **ideography**.

BEGRIFFSSCHRIFT,

EINE DER ARITHMETISCHEN NACHGEBILDETE

FORMELSPRACHE

DES REINEN DENKENS.

VON

Dʳ· GOTTLOB FREGE,

PRIVATDOCENTEN DER MATHEMATIK AN DER UNIVERSITÄT JENA.

HALLE ᴬ/S.

VERLAG VON LOUIS NEBERT.

1879.

Section 3: Basic Definitions

Logical Symbols

The following types of logical symbols are relevant for our analyses:

- **Logical operators (connectives)** equivalent to the ones defined in propositional logic: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- The **quantifier** symbols: $\forall$ (universal quantifier), $\exists$ (existential quantifier)
- An infinite set of variables: x, y, z, etc.[6]
- Parentheses '()' and brackets '[]'[7]

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[6]This set is called *Var* in Zimmermann & Sternefeld (2013), p. 244.

[7]Beware: In the propositional logic notation, we used parentheses '()' for disambiguating the reading of a propositional logic expression as in (p $\rightarrow$ q) $\wedge$ q. However, in the predicate logic notation, parentheses can also have a different function (see below).

Logical Symbols: Quantifiers

"Standard predicate logic makes use of two quantifier symbols: the **Universal Quantifier** $\forall$, and the **Existential Quantifier** $\exists$. As the mathematical examples [below] illustrate, these quantifier symbols **must introduce a variable**, and this variable is said to be bound by the quantifier."

Kroeger (2019) Analyzing meaning, p. 69.

Examples:

For all x it is the case that x plus x equals x times two.
There is some y for which y plus four equals y divided by three.

Quantifier notation:

$\forall$ x [x+x = 2x]
$\exists$ y [y+4 = y/3]

Note: The square brackets are used here to illustrate the formulation that the quantifier scopes over.

Non-Logical Symbols

The following types of non-logical symbols are relevant for our analyses:

- **Predicate symbols**: these are typically given as upper case letters, and reflect relations between n elements, where $n \geq 0$, and $n \in \mathbb{N}$ (i.e. natural numbers). [8]
- **Function symbols**: these are typically given with lower case letters (f, g, etc.), and take n variables as their arguments (similar to predicates), e.g. $f(x)$, $f(x,y)$, etc.

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[8]Zimmermann & Sternefeld (2013), p. 245 denote the set of all n-place predicates of a so-called *predicate logic lexicon* or *language L* as $PRED_{n,L}$.

Non-Logical Symbols: Predicates

Predicate symbols: these are typically given as upper case letters, and reflect relations between n elements, where $n \geq 0$, and $n \in \mathbb{N}$ (i.e. natural numbers). These are also called **n-ary** or **n-place predicate symbols**: $P(x)$, $P(x, y)$, $Q(x, y)$, etc.

Examples:

x snores
x is honest
x sees y
x gives y z

Predicate notation:

P(x)≡ SNORE(x)
Q(x)≡ HONEST(x)
R(x,y)≡ SEE(x,y)
S(x,y,z)≡ GIVE(x,y,z)

The single upper case letter notation is used by Zimmermann & Sternefeld (2013), the all capital notation is used by Kroeger (2019). Yet another notation involving primes (e.g. snore'was used earlier in the lecture following Müller (2019). In the following we will use the notation by Kroeger.

Non-Logical Symbols: Functions

Function symbols are different from predicates since they do not denote a relation between the variables, but they **map the variables to unique values**. Importantly, a function with $n = 0$, i.e. zero valence, is called a **constant symbol** and denotes for example an individual or object.

Examples: Function notation:

Socrates   s
Paris   p
a crocodile   c
father of x   f(x)

Note: *s*, *j*, *p*, and *c* are *constant symbols* here, i.e. strictly speaking zero valence functions, while $f(x)$ is a monovalent function. It is important to realize that while lower case letters are used for both *constant symbols* and *variables* (i.e. *x*), they represent different elements of predicate logic. The convention here is to use the *first letter of the respective name in lower case* as a constant symbol, while variables start at *x*.

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Section 4: Translating to Predicate Logic

N-place Predicates

Predicates with *n* arguments (*n-place*, *n-ary*) are simply dealt with by adding the *constant symbols* in parentheses of the predicate. The order of arguments is assumed to be the same as in the sentence.

(5) *Henry VIII snores.*   SNORE(h)

(6) *Socrates is a man.*   MAN(s)[9]

(7) Abraham admired Victoria.   ADMIRE(a,v)[10]

(8) *Maria dio a Juan un libro.* DAR(m,j,l)

(9) *Jocasta is the mother of Oedipus.*   MOTHER_OF(j,o)[11]

_____

[9]Remember that we said in the lectures on headedness, that sentences with the copular are a tricky case. Note that predicate logic treats the noun *man* here as the main predicate, rather than the copular.

[10]Inflections are not considered by predicate logic, hence, different persons, tenses, etc. are not reflected in the predicate logic formulation.

[11]While just the phrase *mother of oedipus* would have to be construed as a function, e.g. *f(o)*, the phrase *is mother of oedipus* is represented by a predicate.

Predicates and Logical Operators

We can straightforwardly use **logical operators** in connection with **predicates**. For disambuiguating the scope of logical operators (e.g. negation below) we will use square brackets, rather than normal parentheses.

(10) *Abraham Lincoln was tall and homely.* $TALL(a) \wedge HOMELY(a)$

(11) *Socrates was a bright man.* $BRIGHT(s) \wedge MAN(s)$

(12) *Joe is neither honest nor competent.* $\neg[HONEST(j) \vee COMPETENT(j)]$[12]

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[12]In the last example, Kroeger (2019) p. 67 uses parentheses for both. Also, note that we here have to use the inclusive or $\vee$, rather than the exclusive XOR, since only the former allows for both predicates to be true, which is needed to negate them in a *neither...nor* statement.

Predicates of Predicates

Predicates can also be **embedded** into other predicates.

(13) *Henry thinks that*
      Anne is beautiful.   THINK(h, BEAUTIFUL(a))

(14) *Susan wants to marry Ringo*   WANT(s, MARRY(s,r))

(15) *If you are honest, people will perceive you as competent.*
      HONEST(y) → PERCEIVE(p, COMPETENT(y))

Predicates and Quantifiers

Importantly, formulating predicates which involve **quantifications** requires the usage of particular logical operators, since quantifiers require variables, and the variables then need to be further linked to predicates via logical operators.

(16) *All students are weary.* $\forall x[STUDENT(x) \rightarrow WEARY(x)]$
      lit. "For all x it is the case that if x is a student, then x is weary."

(17) *Some men snore.* $\exists x[MAN(x) \wedge SNORE(x)]$
      lit. "There exists some x for which it is the case that x is a man and x snores."[13]

(18) No crocodile is warm-blooded. $\neg \exists x[CROCODILE(x) \wedge WARM\text{-}BLOODED(x)]$
      lit. "It is not the case that there is some x for which x is a crocodile and x is warm-blooded."

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[13]Note that while the plural *men* suggests that we are talking about 2 or more individuals, the predicate logic formulation is valid for 1 or more individual(s).

Multi-Valent Predicates and Quantifiers

In the case of **multi-valent predicates** being combined with **quantifiers**, we typically have a combination of *variables* and *constant symbols* as arguments of the predicates. Indefinite noun phrases are typically translated using the existential quantifier.

(19) *Mary knows all the professors.*
∀x[PROFESSOR(x)→KNOW(m,x)]
lit. "For all x it is the case that if x is a professor, then Mary knows x."

(20) *Susan married a cowboy.*
∃x[COWBOY(x)∧MARRY(s,x)]
lit. "For some x it is the case that x is a cowboy and Susan married x."[14]

(21) *Ringo lives in a yellow submarine.*
∃x[YELLOW(x)∧SUBMARINE(x)∧LIVE_IN(r,x)]
lit. "For some x it is the case that x is yellow, and x is a submarine, and that Ringo lives in x."

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[14]Alternatively, we could drop the indefinite determiner and formulate just MARRY(s,c). However, this is less precise and hence dispreferred.

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Multi-Valent Predicates and Quantifiers

"When a quantifier combines with another quantifier, with negation, or with various other elements [...], it can give rise to ambiguities of scope."

Kroeger (2019). Analyzing meaning, p. 72.

(22) *Some man loves every woman.*

i. ∃x[MAN(x)∧(∀y[WOMAN(y)→LOVE(x,y)])]
lit. "Fore some x it is the case that x is a man and [for all y it is the case that y is a woman and x loves y]."

ii. ∀y[WOMAN(y)→(∃x[MAN(x)∧LOVE(x,y)])]
lit. "For all y it is the case that if y is a woman then there is an x which is a man and x loves y."

(23) *All that glitters is not gold.*

i. ∀x[GLITTER(x)→ ¬GOLD(x)]
lit. "For all x it is the case that if x glitters then x is not gold."

ii. ¬∀x[GLITTER(x)→GOLD(x)]
lit. "It is not the case for all x that if x glitters then x is gold."

Note: In the first case the ambiguity is between whether the existential quantifier scopes over the universal quantifier, or the other way around. In the second example the ambiguity is whether the negation scopes over the universal quantifier or the other way around.

Section 5: Inference and Predicate Logic

Universal Instantiation

We can now translate the classical types of inferences (which are not covered by prepositopnal logic) into predicate logic notation. Below is a classic inference called **universal instantiation**. By using a variable x bound by the universal quantifier (Premise 1), and then specifiyng this variable as a constant symbol (Premise 2), we adhere to a valid pattern of inference.

(24) Premise 1: **All** *men are mortal.* $\forall x[\text{MAN}(x) \rightarrow \text{MORTAL}(x)]$
       Premise 2: *Socrates is a man.* $\text{MAN}(s)$

       Conclusion: *Therefore, Socrates is mortal.* $\text{MORTAL}(s)$

Existential Generalization

Another classic example is the so-called **existential generalization**. By asserting that two predicates are true for the same constant symbol (premise 1 and premise 2), we can generalize that there has to be a variable x for which both predicates hold.

(25)   Premise 1: *Arthur is a lawyer.*   LAWYER(a)
        Premise 2: *Arthur is honest.*   HONEST(a)
        _____

        Conclusion: *Therefore, **some (= at least one)** lawyer is honest.*
        $\exists x[\text{LAWYER}(x) \wedge \text{HONEST}(x)]$

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Section 6: Evaluating Predicate Logic Expressions

Model Theory

"In order to develop and test a set of interpretive rules [...] it is important to provide very explicit descriptions for the test situations. As stated above, this kind of **description of a situation is called a model**, and must include two types of information: (i) **the domain**, i.e., the set of all individual entities in the situation; and (ii) the **denotation sets for the basic vocabulary items** [constant symbols, predicates] in the expressions being analyzed."

Kroeger (2019). Analyzing meaning, p. 240.

Example Domain

Let us assume our example **domain**, i.e. the so-called **universal set U** is a set of exactly three individuals that we can then use in further more complicated predicate logic expressions.

U = {King Henry VIII, Anne Boleyn, Thomas Moore}

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Example Denotations

Let us further assume the **denotation sets** of three predicates and three constant symbols. These denotation sets specify which individuals of U a particular expression can possibly denote.

⟦MAN⟧ = {King Henry VIII, Thomas Moore}
⟦WOMAN⟧ = {Anne Boleyn}
⟦SNORE⟧ = {King Henry VIII}
⟦h⟧ = {King Henry VIII}
⟦a⟧ = {Anne Boleyn}
⟦t⟧ = {Thomas Moore}

Example Model Evaluation

Based on our **example model**, consisting of the example domain and the example universal set, we can now evaluate the truth values of predicate logic expressions. One-place predicates are evaluated by whether the constant symbol is a member of the denotation set of the predicate. Logical operators are evaluated the same way as in propositional logic. Quantifiers are evaluated according to subset relations.

See Kroeger (2019). Analyzing meaning, p. 241.

	English sentence	logical form	interpretation	truth value
a.	*Thomas More is a man.*	MAN(t)	Thomas More $\in$ ⟦MAN⟧	T
b.	*Anne Boleyn is a man or a woman.*	MAN(a) ∨ WOMAN(a)	Anne Boleyn $\in$ (⟦MAN⟧∪⟦WOMAN⟧)	T
c.	*Henry VIII is a man who snores.*	MAN(h) ∧ SNORE(h)	Henry VIII $\in$ (⟦MAN⟧∩⟦SNORE⟧)	T
d.	*All men snore.*	∀x[MAN(x) → SNORE(x)]	⟦MAN⟧⊆⟦SNORE⟧	F
e.	*No women snore.*	¬∃x[WOMAN(x) ∧ SNORE(x)]	⟦WOMAN⟧∩⟦SNORE⟧ = ∅	T

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Thank You.

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