



Semantics & Pragmatics SoSe 2022

Lecture 7: Formal Semantics IV (Type Theory)



Overview

Section 1: Recap of Lecture 6

Section 2: Beyond Standard Logic
Semantic Compositionality

Section 3: The Theory of Types
Recursive Definition of Types
Semantic Composition in Natural Languages

Section 4: Syntax of Type-Theoretic Languages
The Vocabulary
The Syntax: Recursive Definition

Section 5: Semantics of Type-Theoretic Languages

Summary

References



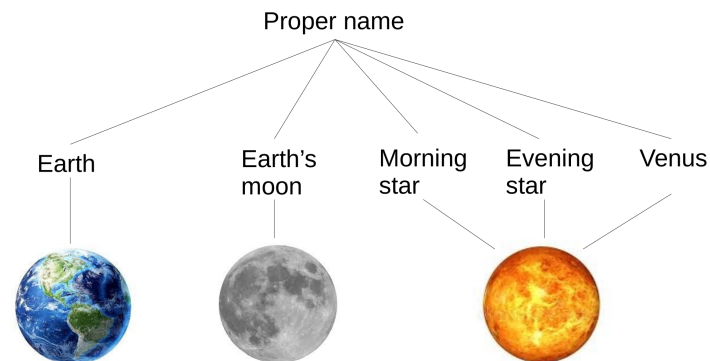
Section 1: Recap of Lecture 6



Frege's Original Statement

Sprache folgend zuschreiben möchte. Wenn man z. B. alle Begriffe, unter welche nur Ein Gegenstand fällt, unter einen Begriff sammelt, so ist die Einzigkeit Merkmal dieses Begriffes. Unter ihn würde z. B. der Begriff „Erdmond,“ aber nicht der sogenannte Himmelskörper fallen. So kann man einen Begriff unter einen höhern, so zu sagen einen Begriff zweiter Ordnung fallen lassen. Dies Verhältniss ist aber nicht mit dem der Unterordnung zu verwechseln.

[...] If you collect, for instance, all notions [*Begriffe*], which have the property of denoting just a single object, under another notion, then the uniqueness is the property of this other notion. For example, the notion “earth’s moon” would fall under this other notion, but not the celestial object itself. Hence, you can have a given notion fall under another notion, a **notion of second order** so to speak. [...]



Frege (1884), p. 65 (paragraph 53).

Section 1: Recap of Lecture 6

Section 2: Beyond Standard Logic

Section 3: The Theory of Types

Section 4: Syntax of Type-Theoretic Languages

Section 5: Semantics of Type-Theoretic Languages

Summary

References



Beyond Predicate Logic

We have seen that predicate logic is an extension of propositional logic, by introducing predicates and quantifiers. **Predicate logic** might itself be superseded by another logical system, called **second-order logic**.

Gamut, L.T.F (1991). Volume 1, p. 168.

Take the following English sentences:

- (1) Mars is red.
- (2) Red is a color.
- (3) Mars has a color.
- (4) John has at least one thing in common with Peter.

How can we translate these into logical expressions?

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Second-Order Predicates

To circumvent this discrepancy, we can construe the predicate *x is a color* not as a property, but as a **property of properties**. \mathcal{C} then represents a so-called **second-order property**, i.e. a **second-order predicate** over the first-order predicate *x is red*.

Instead of

(5) $\mathcal{C}r$ ($\mathcal{C}x$: *x is a color*, r : *red*),

we then get

(6) $\mathcal{C}R$ ($\mathcal{C}X$: *X is a predicate with the property of being a color*, Rx : *x is red*)

Note: We introduce **two** new sets of symbols here compared to standard predicate logic, a) the set of second-order predicates

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Second-Order Logic

A **second-order logic** language L' is then an extension to a standard predicate logic language L , then also referred to as **first-order logic** language, by adding second-order predicates to L .

Further Examples:

- (7) $\exists X(CX \wedge Xm)$ (English sentence: “Mars has a color.”)
- (8) $\exists X(Xj \wedge Xp)$ (English sentence: “John has at least one thing in common with Peter.”)
- (9) $\exists \mathcal{X}(\mathcal{X}R \wedge \mathcal{X}G)$ (English sentence: “Red has something (a property) in common with green.”)

Section 1: Recap of Lecture 6

Section 2: Beyond Standard Logic

Section 3: The Theory of Types

Section 4: Syntax of Type-Theoretic Languages

Section 5: Semantics of Type-Theoretic Languages

Summary

References



Vocabulary (special to Second-Order Logic)

The vocabulary extensions to fit **second-order logic requirements** are:

- ▶ A (potentially infinite) supply of **first-order predicate variables** (e.g. X, Y, Z , etc.), which are necessary to quantify over first-order predicates,
- ▶ a (potentially infinite) supply of **second-order predicate constants** (e.g. $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc.).

If we wanted to take it even at a higher-order level we could also have:

- ▶ a (potentially infinite) supply of **second-order predicate variables** (e.g. $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, etc.) to stand in for second-order predicates.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



The Syntax: Recursive Definition

Given the vocabulary of L we then define the following clauses to create formulas of L :

- (i) If A is an n -ary **first-order** predicate letter/constant in L , and t_1, \dots, t_n are individual terms in L , then At_1, \dots, t_n is an (atomic) formula in L ;
- (ii) If X is a [**first-order**] predicate variable and t is an individual term in L , then Xt is an atomic formula in L ;
- (iii) If \mathcal{A} is an n -ary **second-order** predicate letter/constant in L , and T_1, \dots, T_n are **first-order unary** predicate constants, or predicate variables, in L , then $\mathcal{A}T_1, \dots, T_n$ is an (atomic) formula in L ;
- (iv) If ϕ is a formula in L , then $\neg\phi$ is too;
- (v) If ϕ and ψ are formulas in L , then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are too.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



The Syntax: Recursive Definition

Given the vocabulary of L we then define the following clauses to create formulas of L :

- (vi) If x is an individual variable ϕ is a formula in L , then $\forall x\phi$ and $\exists x\phi$ are also formulas in L ;
- (vii) If X is a [first-order] predicate variable, and ϕ is a formula in L , then $\forall X\phi$ and $\exists X\phi$ are also formulas in L ;
- (viii) Only that which can be generated by the clauses (i)-(vii) in a finite number of steps is a formula in L .

Gamut, L.T.F (1991). Volume 1, p. 170.

Note: In the above clauses (i) and (ii), the word “term” is used, which has not been defined by us before. In the context here, suffices to say that it includes both constants and variables (of constants), i.e. a, b, c , etc. and x, y, z , etc.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Examples of **Valid** and **Invalid** Formulas

Formula	Rule Applied
Aa ✓	(i)
Ax ✓	(i)
Axy ✓	(i)
Xa ✓	(ii)
Xx ✓	(ii)
$\mathcal{A}A$ ✓	(iii)
$Xa \rightarrow \neg Xb$ ✓	(ii), (iv) and (v)
$\forall X \forall x (Xa \rightarrow Axy)$ ✓	(i), (ii), (v), (vi), and (vii)
x ✗	—
X ✗	—
Xab ✗	—
$\forall (Xa)$ ✗	—

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Section 2: Beyond Standard Logic



Beyond Standard Logic

Throughout the last lectures we have seen that propositional logic is superseded by **first-order** predicate logic, while this is in turn superseded by **second-order** predicate logic. Propositional logic and predicate logic are typically grouped together as **Standard Logic**.

As we move from propositional to predicate logic, and then beyond standard logic, we increasingly tease apart sentences into their **component parts**.

Section 1: Recap of Lecture 6

Section 2: Beyond Standard Logic

Section 3: The Theory of Types

Section 4: Syntax of Type-Theoretic Languages

Section 5: Semantics of Type-Theoretic Languages

Summary

References



Example

All animals that live in the jungle have a color.

Propositional logic:

p

First-order predicate logic:

$$\forall x((Ax \wedge Jx) \rightarrow Cx)$$

Translation key: Ax : x is an animal; Jx : x lives in the jungle; Cx : x has a color.

Second-order predicate logic:

$$\forall x(\exists X((\mathcal{A}X \wedge Xx) \wedge Jx) \rightarrow \exists Y(Yx \wedge \mathcal{C}Y))$$

Translation key: $\mathcal{A}X$: x is a property (type of animal) which has the property of being an animal; Jx : x lives in the jungle; $\mathcal{C}X$: X is a property (a particular color) which has the property of being a color.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



The Problem of Semantic Compositionality

However, in both first- and second-order logic, there are still no tools to get to grips with frequent **compositional structures** in natural language:

- ▶ **adjective-noun** combinations
- ▶ **adverb-verb** combinations
- ▶ **article-noun** combinations
- ▶ **preposition-NP** combinations
- ▶ etc.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Example: Adjectives and Nouns

Adjectives and nouns frequently – but not always – combine to yield compound expressions in which the compound meaning is a combination of the individual meanings (e.g. *pink elephant*).

(10) *Jumbo is a pink elephant.*

Predicate Logic: $E_j \wedge P_j$

(11) *Jumbo is a reliable elephant.*

Predicate Logic: $E_j \wedge R_j$

(12) *Jumbo is a small elephant.*

Predicate Logic: $E_j \wedge S_j$

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Discussion Point

Would you consider these pairs of English sentences as equivalent in their meaning?

- ▶ Jumbo is a pink elephant/Jumbo is pink and Jumbo is an elephant.
- ▶ Jumbo is a reliable elephant/Jumbo is reliable and Jumbo is an elephant.
- ▶ Jumbo is a small elephant/Jumbo is small and Jumbo is an elephant.

Why? Why not?

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Adjectives and Nouns

In the first case, the predicate logic expression might reflect the English sentence correctly. In the last example, namely, when a *relative adjective* (e.g. small, big, fast, slow, etc.) is used, the English sentence is misrepresented, since Jumbo might be small for an elephant, but not small in general. The adjective here modifies the noun and results in a new predicate *small elephant*.

Gamut (1991), Volume 2, p. 77.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References

- (13) *Jumbo is a pink elephant.*
Predicate Logic: $E_j \wedge P_j$
- (14) *Jumbo is a reliable elephant.*
Predicate Logic: $E_j \wedge R_j$
- (15) *Jumbo is a small elephant.*
Predicate Logic: $E_j \wedge S_j$



Example: Adverbs and Verbs

Adverbs and verbs likewise combine to yield compound expressions. Again, the available predicate logic translation falls short of the actual meaning of *walking quickly*. The adverb modifies the verb and creates a new predicate.

(16) *Jumbo is walking quickly.*

(17) *Jumbo is walking and Jumbo is quick.*

Predicate logic: $W_j \wedge Q_j$

Gamut (1991), Volume 2, p. 77.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Types of Expressions

There is a long (potentially infinite) list of **types of expressions** which we might want to represent in our logical language in order to capture the different combinatorial possibilities we find in natural languages.

Kind of expression

Individual expression
One-place first-order predicate
Two-place first-order predicate
Three-place first-order predicate
Sentence
Sentential modifier
Function
Predicate modifier
One-place second-order predicate
Two-place second-order relation
etc.

Examples

John, Jumbo
walks, red, loves Mary
loves, lies between Amsterdam and
lies between (and)
John walks, John loves Mary
not
the father of
quickly, beautifully, fast
is a color
is a brighter color than
etc.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Section 3: The Theory of Types



The Theory of Types

How can we represent this **potentially infinite number of expressions** while preserving their *internal structure* and *combinatorial relationships*? – A logical system developed to fit this requirement is the so-called **theory of types** which was developed by Bertrand Russell as a remedy for paradoxes encountered in set theory.

Gamut (1991), Volume 2, p. 78.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

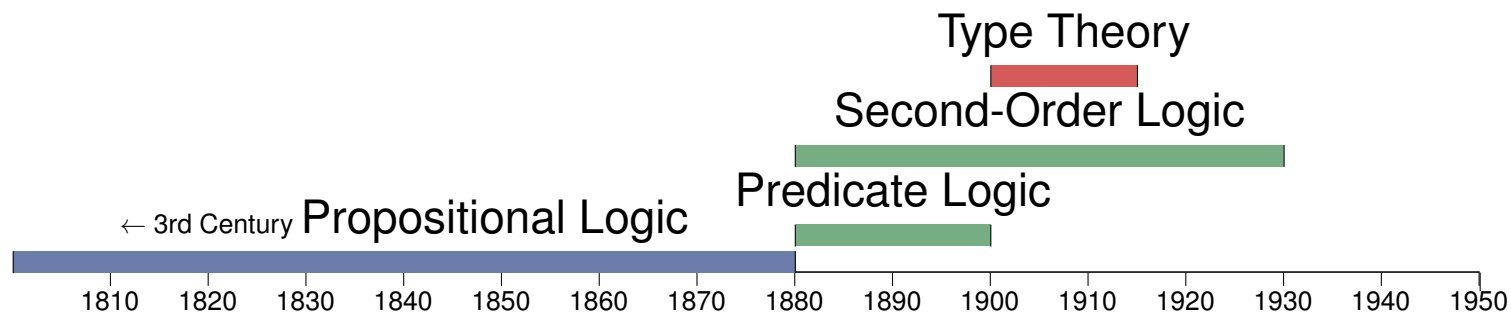
Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

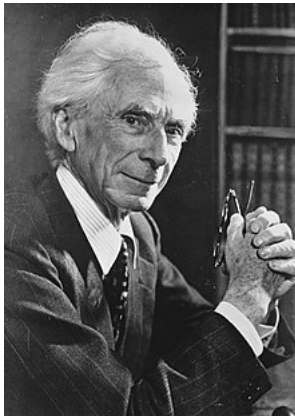
Summary

References





Russell's Paradox



The Contradictions.

(1) The oldest contradiction of the kind in question is the *Epimenides*. Epimenides the Cretan said that all Cretans were liars, and all other statements made by Cretans were certainly lies. Was this a lie? The simplest form of this contradiction is afforded by the man who says “I am lying;” if he is lying, he is speaking the truth, and vice versa.

(2) Let w be the class of all those classes which are not members of themselves. Then, whatever class x may be, “ x is a w ” is equivalent † to “ x is not an x .” Hence, giving to x the value w , “ w is a w ” is equivalent to “ w is not a w .”

Russell (1908), p. 222.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Historical Note: Russell's Paradox

- ▶ There are sets which contain themselves, e.g. $\mathbb{U} = \{x \mid x = x\}$, which is called the *universal set*, and for which it holds that $\mathbb{U} \in \mathbb{U}$. It contains everything that exists, since everything is equal to itself.
- ▶ However, most familiar sets do not contain themselves, e.g. the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, for which $\mathbb{N} \notin \mathbb{N}$, since a set is not a natural number.
- ▶ We thus might want to define the set of entities which are not members of themselves $\mathbb{R} = \{x \mid x \notin x\}$. A paradoxical question arises: is \mathbb{R} a member of itself or not?
- ▶ Consider the so-called *Barber Paradox* as a more intuitive illustration: *Assume a barber shaves all those who do not shave themselves. – Does the barber shave himself?*

Gamut (1991), Volume 2, p. 78.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Historical Note: Russell's Solution

Each entity is assigned a **type** which represents the “level” of the logical language where the logical expression is located at. For instance, a natural number n is an individual entity. The set of natural numbers \mathbb{N} , on the other hand, is of the type “set of individual entities”.

The membership relation is then defined to only apply to logical expressions which are **exactly one level apart**. Hence, $n \in \mathbb{N}$ is a valid expression in the logical language of types, while $\mathbb{N} \in \mathbb{N}$ is not.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Type Theory Applied to Language

“Linguistic expressions are classified into their **semantic types** according to the kind of denotation they have. The two most basic denotation types are **type e**, the type of **entities**, and **type t**, the type of **truth values**.”

Kearns (2011). Semantics, p. 57.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References

Type of expression	Type of extension	Semantic type	Example
proper name	individual (entity)	e	$[[\text{Paul}]]_s = \text{Paul McCartney}$
...
sentence	truth value	t	$[[\text{Paul is happy}]]_s \in \{0, 1\}$



Definition: The Syntax of Types

For the set of types \mathbb{T} we define that:

- (i) $e, t \in \mathbb{T}$,
- (ii) if $a, b \in \mathbb{T}$, then $\langle a, b \rangle \in \mathbb{T}$,
- (iii) nothing is an element of \mathbb{T} except on the basis of clauses (i) and (ii).

Gamut (1991), Volume 2, p. 79.

Note: a and b above are variables which stand in for all kinds of types. This means we can create an infinite number of types by recursively applying clause (ii). For example:

Applying (ii) to $a = e$ and $b = t$ yields $\langle e, t \rangle$

Applying (ii) to $a = \langle e, t \rangle$ and $b = t$ yields $\langle \langle e, t \rangle, t \rangle$

Applying (ii) to $a = e$ and $b = \langle \langle e, t \rangle, t \rangle$ yields $\langle e, \langle \langle e, t \rangle, t \rangle \rangle$

etc.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Examples of **Valid** and **Invalid** Types

e ✓

t ✓

$\langle e, t \rangle$ ✓

$\langle t, e \rangle$ ✓

$\langle t, \langle t, e \rangle \rangle$ ✓

$\langle \langle t, \langle t, e \rangle \rangle, t \rangle$ ✓

et ✗

e, t ✗

$\langle e, e, t \rangle$ ✗

$\langle e, \langle e, t \rangle \rangle$ ✗

Note: The usage of left and right angled brackets as defined by clause (ii) results in a **strict binarization** of the internal structure of types, i.e. at each level of embedding we always have an **ordered pair** of more basic types.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Definition: Functional Application

How do we derive one type of expression from another?

“[...] if α is an expression of type $\langle a, b \rangle$ and β is an expression of type a , then $\alpha(\beta)$ is of type b .”

Gamut (1991), Volume 2, p. 79.

Examples

If $\alpha = \langle e, t \rangle$ and $\beta = e$ then $\alpha(\beta) = t$.

If $\alpha = \langle \langle e, t \rangle, \langle e, t \rangle \rangle$ and $\beta = \langle e, t \rangle$ then $\alpha(\beta) = \langle e, t \rangle$.

If $\alpha = \langle t, \langle t, e \rangle \rangle$ and $\beta = t$ then $\alpha(\beta) = \langle t, e \rangle$.

However,

If $\alpha = \langle t, \langle t, e \rangle \rangle$ and $\beta = \langle t, e \rangle$ then $\alpha(\beta)$ is **not defined**.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

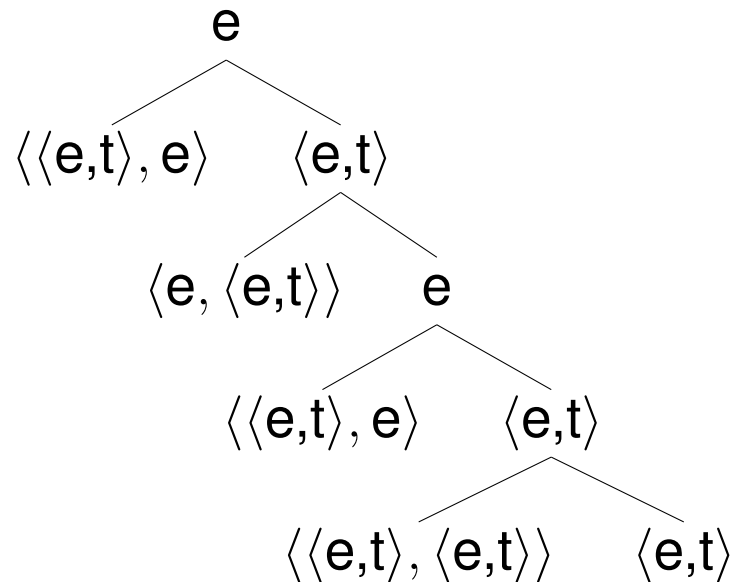
Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Example: Recursive Application



Note: Binarization does **not** mean that there are only a maximum of two components in each overall tuple. Instead there can be infinitely many 2-tuple embeddings. But each individual tuple can only have two components. Hence, we can build more complex semantic types out of the two basic types e and t .

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Functional Application in Natural Language Semantics

“[...] a function binds arguments together into a statement. From this insight, Frege proposed that all **semantic composition** is **functional application**. Functional application is just the combination of a function with an argument.”

Kearns (2011), p. 58.

Note: This means that in a *type-theoretic logical language*, the distinction between predicates and functions is irrelevant. All structure building is based on functional application.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Semantic Types: Individuals

An **individual**, represented by a logical constant of the semantic **type** e , corresponds to, for instance, a proper noun (N) in natural language.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

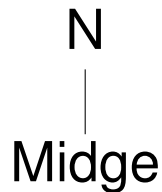
Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

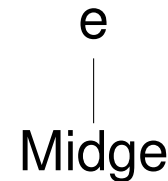
Summary

References

Syntactic Tree



Type-Theoretic Tree

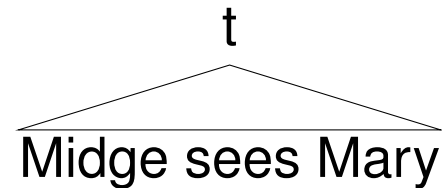
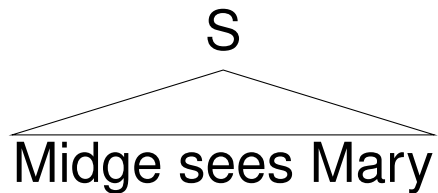


This and the following examples are based on Kearns (2011).



Semantic Types: Sentences

Logical formulae of the semantic **type** t correspond to sentences (S) in natural language.



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

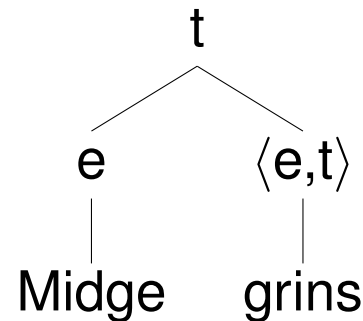
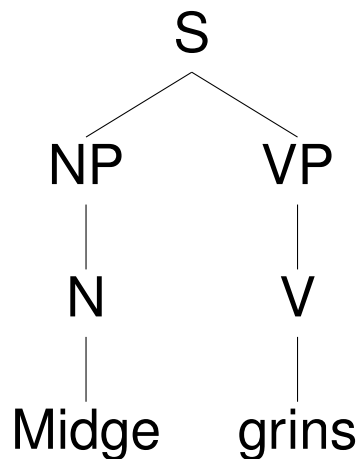
Summary

References



Semantic Types: One-Place Predicates

An **intransitive verb** requires **one argument** to be filled in order to form a full sentence, hence it is of the **type** $\langle e, t \rangle$. Remember that the argument is on the left side of the tuple (ordered pair), hence the component of type *entity* (*e*) is left.



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

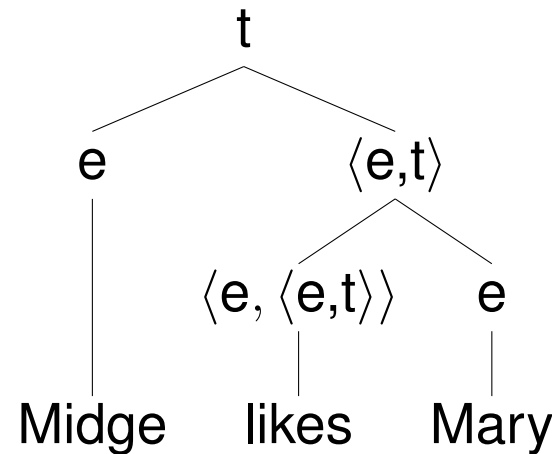
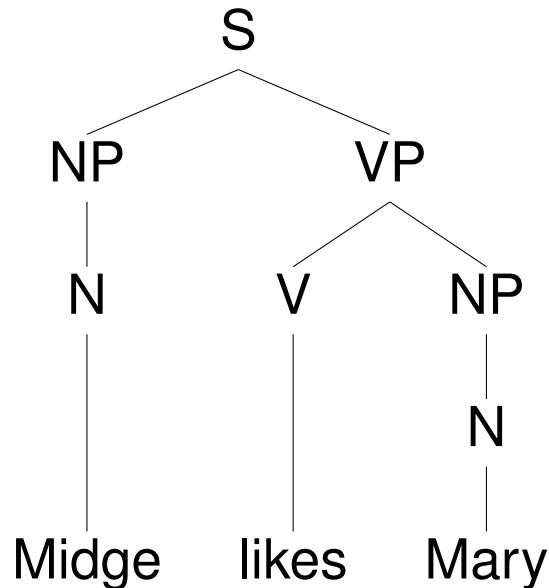
Summary

References



Semantic Types: Two-Place Predicates

A **transitive verb** requires **two arguments** to be filled in order to form a full sentence, hence it is of the **type** $\langle e, \langle e, t \rangle \rangle$.



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

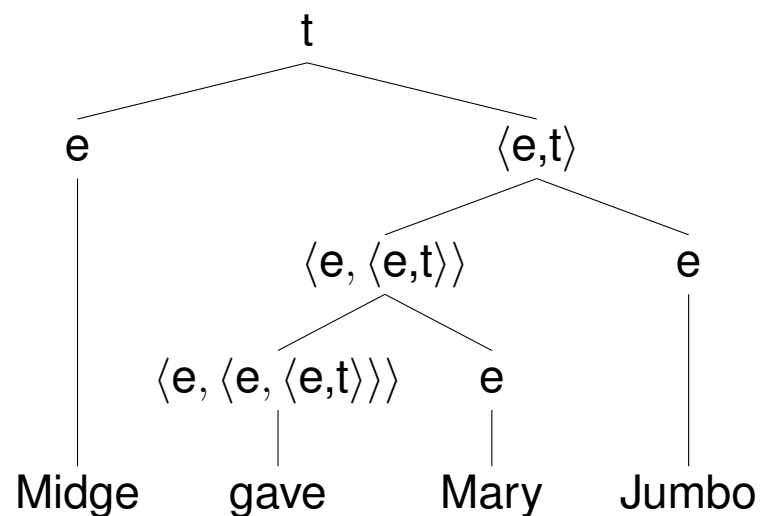
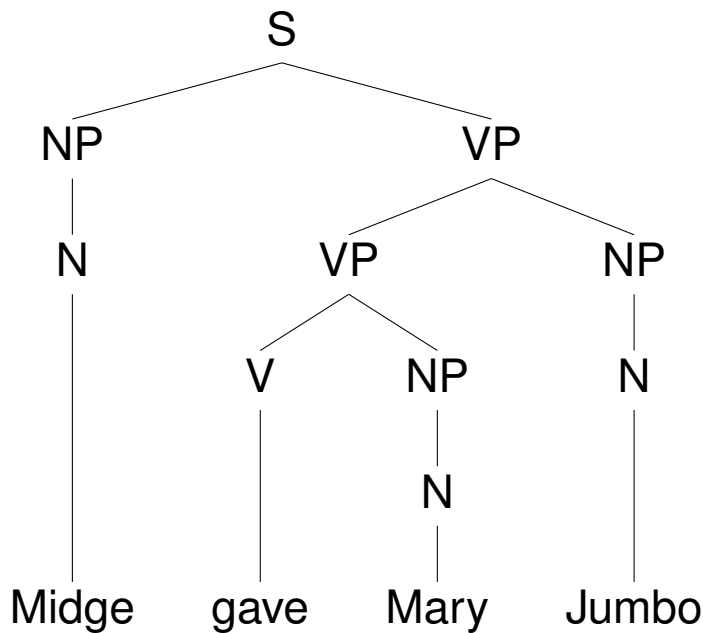
Summary

References



Semantic Types: Three-Place Predicates

A **ditransitive verb** requires **three arguments** to be filled in order to form a full sentence, hence it is of the **type** $\langle e, \langle e, \langle e, t \rangle \rangle \rangle$.



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

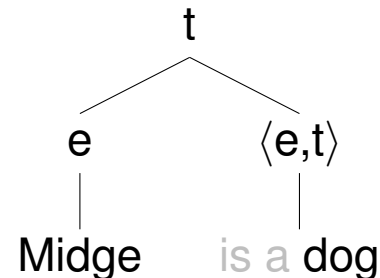
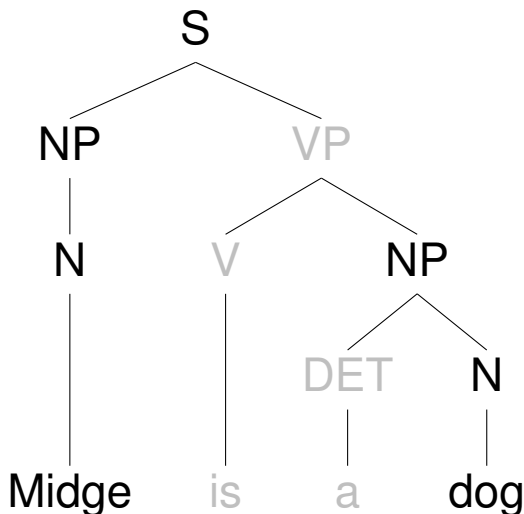
Summary

References



Semantic Types: Common Nouns

Common nouns are of type **type** $\langle e, t \rangle$. This might seem counterintuitive at first sight, but the idea here is that nouns are essentially like **one-place predicates**, in the sense that they require a concrete entity (e) to form a basic existential statement (with a copular) which can be true or false.



Note: This corresponds to the predicate logic formulation D_m , where the copula and the indefinite determiner are dropped. As pointed out earlier, the copula is a controversial case, and the syntactic tree given here assumes that the copula is heading a VP, which is not uncontroversial.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

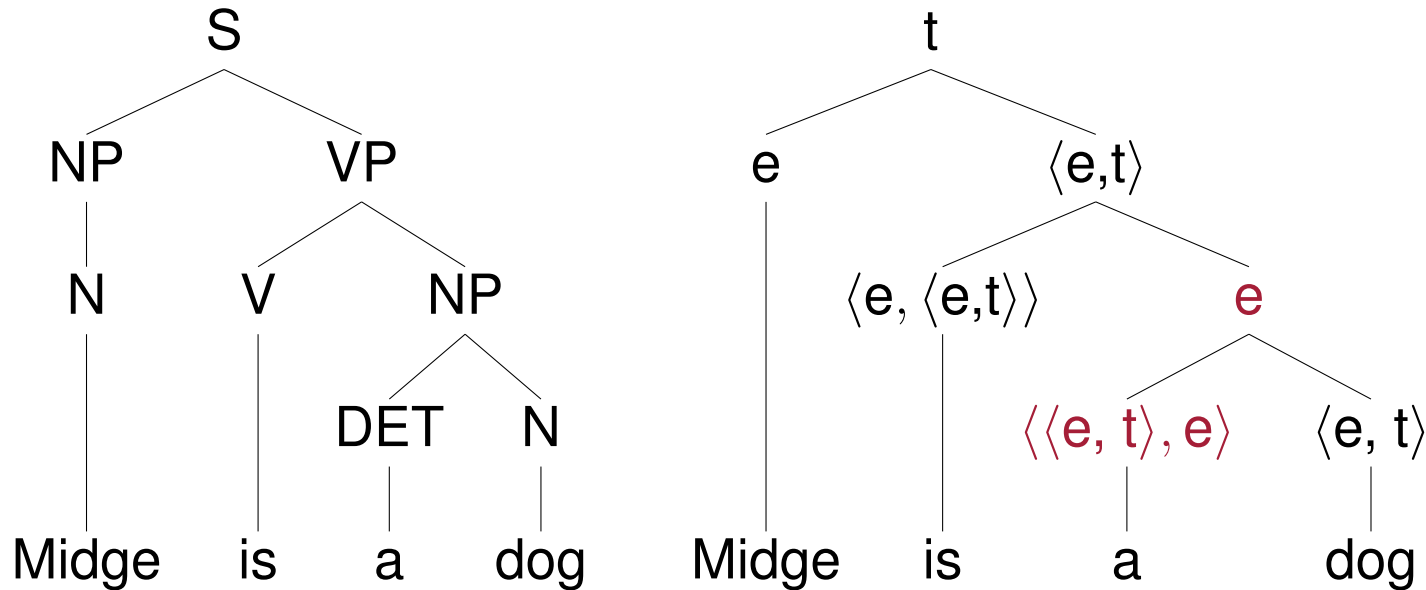
Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Alternative Analysis?



Note: This might seem like a valid alternative, but notice that the type of *a dog* has to be e now, meaning that it is an individual, rather than a set of individuals. So this would break with the fundamental definition that indeterminate expressions have sets as their extensions. Also, the *indefinite determiner* would then be of the same type as the *definite determiner* (see next slide). See also Kearns (2011), p. 149 for a discussion of *specific* and *non-specific* readings of indefinite descriptions.

Section 1: Recap of Lecture 6

Section 2: Beyond Standard Logic

Section 3: The Theory of Types

Section 4: Syntax of Type-Theoretic Languages

Section 5: Semantics of Type-Theoretic Languages

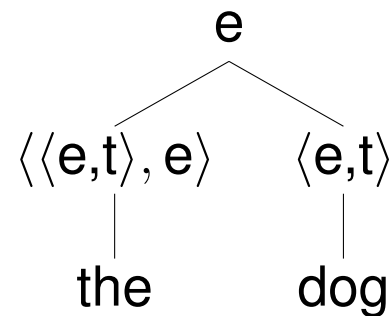
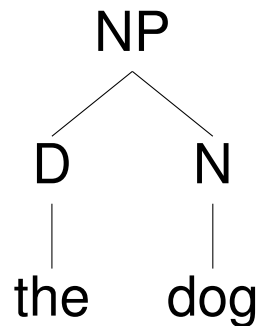
Summary

References



Semantic Types: NPs and Determiners

NPs are of **type e**, i.e. referring to a concrete entity. Note that it follows from this definition and the definition of common nouns above that **determiners** then have to be of **type $\langle\langle e,t \rangle, e\rangle$** .



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

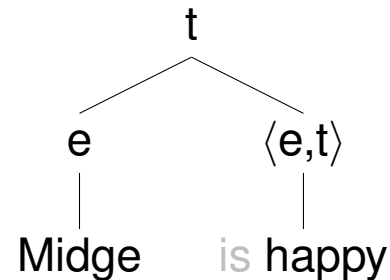
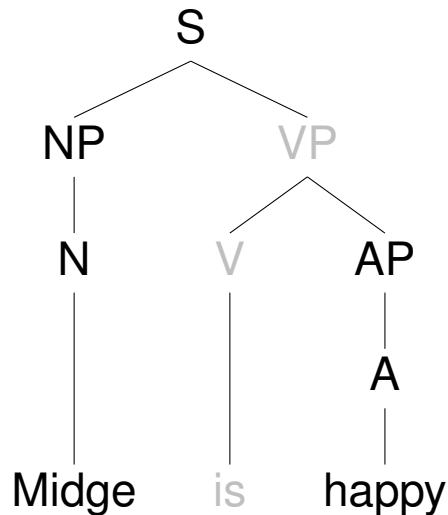
Summary

References



Semantic Types: Adjectives (Existential Statements)

Similar to common nouns, **adjectives** are considered to be of **type** $\langle e, t \rangle$. The same argument applies: they require a concrete entity (e) to form a basic existential statement (with a copular) which can be true or false.



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

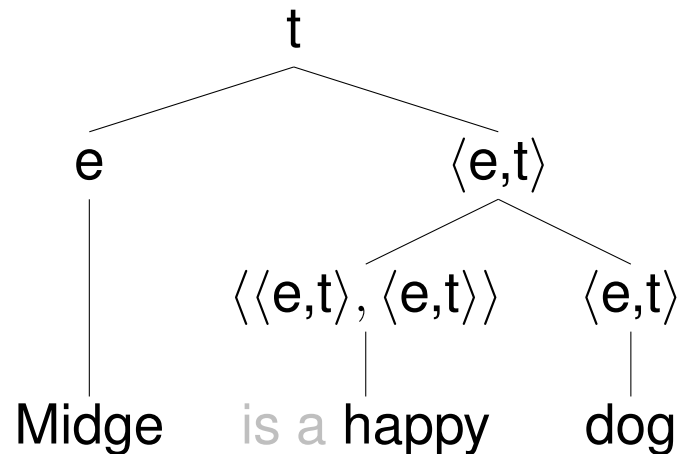
References



Semantic Types: Adjectives (Predicate Modifiers)

On the other hand, adjectives can also serve as **predicate modifiers**. In this case, they have to be defined as being of the type $\langle\langle e,t \rangle, \langle e,t \rangle\rangle$ (same as adverbs below).

Gamut (1991), Volume 2, p. 77.



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

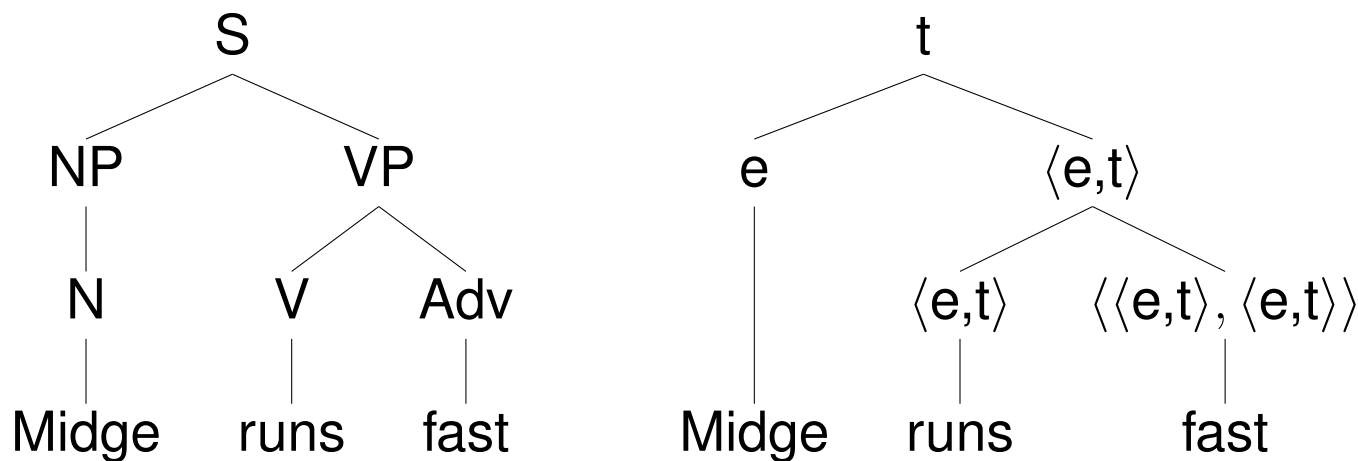
Summary

References



Semantic Types: Adverbs

Adverbs are considered **type** $\langle\langle e,t \rangle, \langle e,t \rangle\rangle$. Note that similar as for determiners, this is a logical consequence of the definition of other types, i.e. the definition of a one-place predicate modified by an adverb.



Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Summary: Types of Expressions

There is a long (potentially infinite) list of **types of expressions** which we might want to represent in our logical language in order to capture the different combinatorial possibilities we find in natural languages.

Type	Kind of expression	Examples
e	Individual expression	<i>John, Jumbo</i>
$\langle e, t \rangle$	One-place first-order predicate	<i>walks, red, loves Mary</i>
$\langle e, \langle e, t \rangle \rangle$	Two-place first-order predicate	<i>loves, sees</i>
$\langle e, \langle e, \langle e, t \rangle \rangle \rangle$	Three-place first-order predicate	<i>lies between (and)</i>
t	Sentence	<i>John walks, John loves Mary</i>
$\langle t, t \rangle$	Sentential modifier	<i>not</i>
$\langle e, e \rangle$	Function (entity to entity)	<i>the father of</i>
$\langle \langle e, t \rangle, \langle e, t \rangle \rangle$	Predicate modifier	<i>quickly, beautifully, fast</i>
$\langle \langle e, t \rangle, t \rangle$	One-place second-order predicate	<i>is a color</i>
$\langle \langle e, t \rangle, \langle \langle e, t \rangle, t \rangle \rangle$	Two-place second-order predicate	<i>is a brighter color than</i>
etc.	etc.	etc.

Based on Gamut (1991), Volume 2, p. 86.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Section 4: Syntax of Type-Theoretic Languages



Vocabulary (Shared)

The shared part of the vocabulary consists of:

- ▶ For every type a , an infinite set VAR_a of variables of type a ;¹
- ▶ the usual connectives $\vee, \wedge, \rightarrow, \leftrightarrow, \neg$;
- ▶ the quantifiers \forall and \exists ;
- ▶ two brackets '(' and ')';
- ▶ the symbol for identity '='.²

Gamut (1991), Volume 2, p. 80.

¹For example, for the type of individuals e we can define an infinite set of variable symbols x, y, z , etc. of this type (just as for predicate logic). Likewise, for the type of a one-place predicate $\langle e, t \rangle$ we can also define an infinite set of one-place first order predicate variables X, Y, Z , etc. (just as for predicate logic), and so on for every type.

²Optional as before, depending on whether we want to use expressions which reflect equations.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Vocabulary (Characteristic)

The characteristic part of the vocabulary for a particular type-theoretic language consists of:

- ▶ for every type a , a (possibly empty) set CON_a^L of constants of type a .³

Note: It is important to keep constants of type a (c_a) and variables of type a (v_a) apart. A notational difference to predicate logic languages is that the type of a constant or variable can be indicated by an index.

³For example, for the type of individuals e we can define an infinite set of constant symbols a, b, c , etc. of this type (just as for predicate logic). Likewise, for the type of a one-place predicate $\langle e, t \rangle$ we can also define an infinite set of one-place first order predicate constants A, B, C , etc. (just as for predicate logic), and so on for every type.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Notation for Variables and Constants

As before, we will use the following notations to distinguish typographically between different variables and constants at different orders:

- ▶ Constants for entities: a, b, c , etc.
- ▶ Variables over entities: x, y, z , etc.
- ▶ First-order predicate constants: A, B, C , etc.
- ▶ Variables over first-order predicates: X, Y, Z , etc.
- ▶ Second-order predicate constants: $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc.
- ▶ (Second-order predicate variables: $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, etc.)⁴

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References

⁴These are just added for completeness here. We generally don't go into orders higher than *two* in exercises and examples.



The Syntax: Recursive Definition

The clauses for the syntax of a **type-theoretic language** are then:

- (i) If α is a variable or a constant of type a in L [i.e. v_a or c_a], then α is an expression of type a in L .
- (ii) If α is an expression of type $\langle a, b \rangle$ in L , and β is an expression of type a in L , then $(\alpha(\beta))$ is an expression of type b in L .
- (iii) If ϕ and ψ are expressions of type t in L (i.e. formulas in L), then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.
- (iv) If ϕ is an expression of type t in L and v is a variable (of arbitrary type a), then $\forall v\phi$ and $\exists v\phi$ are expression of type t in L .
- (v) If α and β are expressions in L which belong to the same (arbitrary) type, then $(\alpha = \beta)$ is an expression of type t in L .
- (vi) Every expression L is to be constructed by means of (i)-(v) in a finite number of steps.

Gamut (1991), Volume 2, p. 81-82.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



First-Order Predicate Logic

- (i) If A is an n -ary predicate letter in the vocabulary of L , and each of t_1, \dots, t_n is a constant or a variable in the vocabulary of L , then At_1, \dots, t_n is a formula in L .
- (ii) If ϕ is a formula in L , then $\neg\phi$ is too.
- (iii) If ϕ and ψ are formulas in L , then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are too.
- (iv) If ϕ is a formula in L and x is a variable, then $\forall x\phi$ and $\exists x\phi$ are formulas in L .
- (v) Only that which can be generated by the clauses (i)-(iv) in a finite number of steps is a formula in L .

Type-Theoretic Logic

- (i) If α is a variable or a constant of type a in L [i.e. v_a or c_a], then α is an expression of type a in L .
- (ii) If α is an expression of type $\langle a, b \rangle$ in L , and β is an expression of type a in L , then $(\alpha(\beta))$ is an expression of type b in L .
- (iii) If ϕ and ψ are expressions of type t in L (i.e. formulas in L), then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.
- (iv) If ϕ is an expression of type t in L and v is a variable (of arbitrary type a), then $\forall v\phi$ and $\exists v\phi$ are expression of type t in L .
- (v) Every expression L is to be constructed by means of (i)-(v) in a finite number of steps.

Section 1: Recap of Lecture 6

Section 2: Beyond Standard Logic

Section 3: The Theory of Types

Section 4: Syntax of Type-Theoretic Languages

Section 5: Semantics of Type-Theoretic Languages

Summary

References



Examples of **Valid** and **Invalid** Expressions

Definition of Types

Assume j is of type e (i.e. representing an entity), x is of type e , A is of type $\langle e, t \rangle$ (i.e. a first order one-place predicate), B is of type $\langle e, \langle e, t \rangle \rangle$ (i.e. a first-order two-place predicate), and \mathcal{C} is of type $\langle \langle e, t \rangle, t \rangle$ (i.e. a second-order one-place predicate).

Expressions

j ✓

A ✓

$A(j)$ ✓

$(B(j))(x)$ ✓ alternative notation: $B(j)(x)$

$\mathcal{C}(B(j))$ ✓

$A(j) \wedge \mathcal{C}(A)$ ✓

$\forall x A(x)$ ✓

Aj ✗

$B(A)$ ✗

$\forall x \mathcal{C}(x)$ ✗

Clause Applied

(i)

(i)

(i) and (ii)

(i) and (ii)

(i) and (ii)

(i), (ii), and (iii)

(i), (ii), and (iv)

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Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Important: *Expressions and Formulas*

Note that in the definitions above, Gamut use the term **expression** instead of *formula* or *sentence* (which were used before in predicate logic). They further specify the difference:

“The inductive definition of the formulas is more complicated than in predicate logic. For what we have to give is a general definition of what is to be an expression of a type $a \in \mathbb{T}$, the **formulas** are then those expressions which are of the particular type t .”

Gamut (1991), Volume 2, p. 81.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Important Difference to Predicate Logic

Assume j of type e refers to Jumbo, m of type e refers to Maya, B is of type $\langle e, \langle e, t \rangle \rangle$, and is reflecting the English two-place predicate “to befriend”. The English sentence *Jumbo befriends Maya* would then be translated as:

Predicate Logic: Bjm

Type-Theoretic Logic: $(B(m))(j)$ or alternatively $B(m)(j)$ ⁵

Some notable points:

- ▶ While the predicate logic formula Bjm combines all three elements B , m , and j together in a single step, in the type-theoretic account, we are bound to strictly **binary functional application**. In the case of two-place predicates, this means that the predicate/function is applied first to one of the arguments, i.e. $B(m)$, and then, in the second step, the outcome is applied to the next argument, i.e. $(B(m))(j)$.
- ▶ Note that the latter account reflects the **binary tree structure** posed in (some) syntactic frameworks, whereas the former has a flat structure.
- ▶ Also, the **order of the arguments is inverted** here. It is generally assumed that the predicate is first applied to the object of the natural language sentence, and then to the subject.

⁵We leave away the outermost brackets here. Gamut (1991), p. 82 suggest to also drop the brackets around $B(m)$ and hence get the alternative simplified writing $B(m)(j)$.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Section 5: Semantics of Type-Theoretic Languages



Truth Valuation

As seen before for other logical languages, the semantic side of type-theoretic languages consists of the **valuation of truth** given a syntactically valid expression.

Example

Assume *walks* is represented by W of type $\langle e, t \rangle$. Further, *Jumbo* is represented by the constant j of type e . We thus have a valid truth-theoretic expression $W(j)$ of type t (i.e. a formula) representing *Jumbo walks*.

To evaluate the truth of $W(j)$ we need to define a set of all relevant entities D (the domain) with members d , and a subset $W \subseteq D$ whose members can be said to *walk* (i.e. $j \in W$). We then define an interpretation function I for which it holds that:

$$I(W)(d) = 1 \text{ iff } d \in W; \text{ and } I(W)(d) = 0 \text{ iff } d \notin W. \quad (1)$$

$I(W)$ is a so-called *characteristic function of W* (over D).

Gamut (1991), Volume 2, p. 83-87.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Truth Valuation

However, the complexity of defining interpretation functions for all kinds of different types of expressions (see table below from Gamut) is beyond the scope of this course.

Gamut (1991), Volume 2, p. 86.

Type	Interpretation
e	Entity
$\langle e, t \rangle$	Function from entities to truth values, i.e. characteristic function
$\langle e, \langle e, t \rangle \rangle$	Function from entities to sets of entities
$\langle e, \langle e, \langle e, t \rangle \rangle \rangle$	Function from entities to functions from entities to sets of entities
t	Truth value
$\langle t, t \rangle$	Function from truth values to truth values
$\langle e, e \rangle$	Function from entities to entities
$\langle \langle e, t \rangle, \langle e, t \rangle \rangle$	Function from sets of entities to sets of entities
$\langle \langle e, t \rangle, t \rangle$	Characteristic function of a set of sets of entities
$\langle \langle e, t \rangle, \langle \langle e, t \rangle, t \rangle \rangle$	Function from sets of entities to sets of sets of entities
etc.	etc.

Section 1: Recap of Lecture 6

Section 2: Beyond Standard Logic

Section 3: The Theory of Types

Section 4: Syntax of Type-Theoretic Languages

Section 5: Semantics of Type-Theoretic Languages

Summary

References



Summary



Summary

- ▶ The theory of types enables us to assign a **type** to any kind of **natural language structure**.
- ▶ **Functional applications** of expressions of certain types to one another then enable us to represent the rich **combinatoriality** of natural language structures.
- ▶ **Truth valuations** are possible via particular interpretation functions defined for different types of expressions.

Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



References



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Section 1: Recap
of Lecture 6

Section 2:
Beyond Standard
Logic

Section 3: The
Theory of Types

Section 4: Syntax
of Type-Theoretic
Languages

Section 5:
Semantics of
Type-Theoretic
Languages

Summary

References



Thank You.

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