EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Faculty of Philosophy
General Linguistics

# Semantics & Pragmatics SoSe 2021

## Lecture 8: Formal Semantics V (Lambda Calculus)

**01/06/2021, Christian Bentz**

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# Q&A

## Tutorial 3, Exercise 1

*When creating a translation key for a predicate logic language, isn't it redundant to indicate different types of predicates (with the same starting letter) using indices, when they are of different valency/arity? Example: "x sees y" as $S_1xy$ and "x sleeps" as $S_2x$.*

– Yes, this is true, at least for predicate logic languages the valency/arity of a predicate is already disambiguating here. So in a strict sense it is not necessary.


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# Q&A

## Tutorial 3, Exercise 3 d)

*The expression $\exists x \forall y(Ax \leftrightarrow \exists yBy)$ is said to be a valid formula in the predicate logic language L. Would it also be valid if $\forall y$ was replaced by $\exists y$?*

– Yes. Note that the universal quantifier and the existential quantifier are exactly equivalent from the perspective of the syntactic clause which defines how to create valid formulas in L. So wherever there is a universal quantifier there could also be an existential quantifier instead (and the other way around) without changing the validity of the formula. I added that $\exists$y scopes over By to the solutions. Maybe this caused some confusion here.

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# Q&A

## Lecture 7

- Clarification about *formula*, *sentence*, and *expression*: Defining a **standard predicate logic language** Gamut (1991, Volume 1, p. 77) write: “A sentence is a formula in L which lacks free variables.” Defining a **typed logic language** Gamut (1991, Volume 2, p. 81) write: “[...] formulas are then those expressions which are of the particular type *t*.”

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# Overview


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# Section 1: Recap of Lecture 7

# Example

*All animals that live in the jungle have a color.*

**Propositional logic**:
p

**First-order predicate logic**:
$\forall x((Ax \wedge Jx) \rightarrow Cx)$
Translation key: Ax: x is an animal; Jx: x lives in the jungle; Cx: x has a color.

**Second-order predicate logic**:
$\forall x(\exists X((\mathcal{A}X \wedge Xx) \wedge Jx) \rightarrow \exists Y(Yx \wedge \mathcal{C}Y))$
Translation key: $\mathcal{A}X$: x is a property (type of animal) which has the property of being an animal; Jx: x lives in the jungle; $\mathcal{C}X$: X is a property (a particular color) which has the property of being a color.

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## The Theory of Types

How can we represent this **potentially infinite number of expressions** while conserving their *internal structure* and *combinatorial relationships*? – A logical system developed to fit this requirement is the so-called **theory of types** which was developed by Bertrand Russell as a remedy for paradoxes encountered in set theory.

Gamut (1991), Volume 2, p. 78.

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# Definition: The Syntax of Types

For the set of types $\mathbb{T}$ we define that:

(i) $e, t \in \mathbb{T}$,

(ii) if $a, b \in \mathbb{T}$, then $\langle a, b \rangle \in \mathbb{T}$,

(iii) nothing is an element of $\mathbb{T}$ except on the basis of clauses (i) and (ii).

Gamut (1991), Volume 2, p. 79.

**Note**: *a* and *b* above are variables which stand in for all kinds of types. This means we can create an infinite number of types by recursively applying clause (ii). For example:

Applying (ii) to $a = e$ and $b = t$ yields $\langle e, t \rangle$
Applying (ii) to $a = \langle e, t \rangle$ and $b = t$ yields $\langle \langle e, t \rangle, t \rangle$
Applying (ii) to $a = e$ and $b = \langle \langle e, t \rangle, t \rangle$ yields $\langle e, \langle \langle e, t \rangle, t \rangle \rangle$
etc.

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# Examples of Valid and Invalid Types

e ✓
t ✓
$\langle e, t \rangle$ ✓
$\langle t, e \rangle$ ✓
$\langle t, \langle t, e \rangle \rangle$ ✓
$\langle \langle t, \langle t, e \rangle \rangle, t \rangle$ ✓

et x
$e, t$ x
$\langle e, e, t \rangle$ x
$\langle e, \langle e, t \rangle$ x

Note: The usage of left and right ankled brackets as defined by clause (ii) results in a **strict binarization** of the internal structure of types, i.e. at each level of embedding we always have an **ordered pair** of more basic types.

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# Definition: Functional Application

How do we derive one type of expression from another?

"[...] if $\alpha$ is an expression of type $\langle a, b\rangle$ and $\beta$ is an expression of type $a$, then $\alpha(\beta)$ is of type $b$."

Gamut (1991), Volume 2, p. 79.

## Examples

If $\alpha = \langle e, t\rangle$ and $\beta = e$ then $\alpha(\beta) = t$.
If $\alpha = \langle\langle e, t\rangle, \langle e, t\rangle\rangle$ and $\beta = \langle e, t\rangle$ then $\alpha(\beta) = \langle e, t\rangle$.
If $\alpha = \langle t, \langle t, e\rangle\rangle$ and $\beta = t$ then $\alpha(\beta) = \langle t, e\rangle$.

However,

If $\alpha = \langle t, \langle t, e\rangle\rangle$ and $\beta = \langle t, e\rangle$ then $\alpha(\beta)$ is **not defined**.

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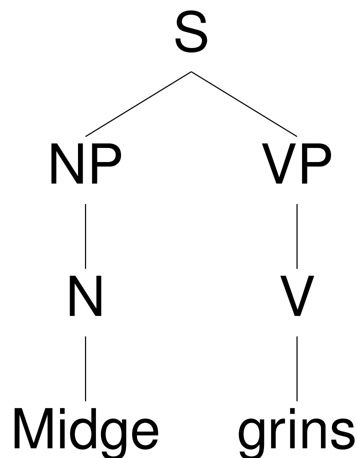
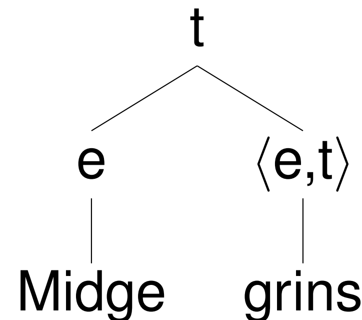
# Semantic Types: One-Place Predicates

An **intransitive verb** requires **one argument** to be filled in order to form a full sentence, hence it is of the **type ⟨e,t⟩**. Remember that the argument is on the left side of the tuple (ordered pair), hence the component of type *entity* (e) is left.

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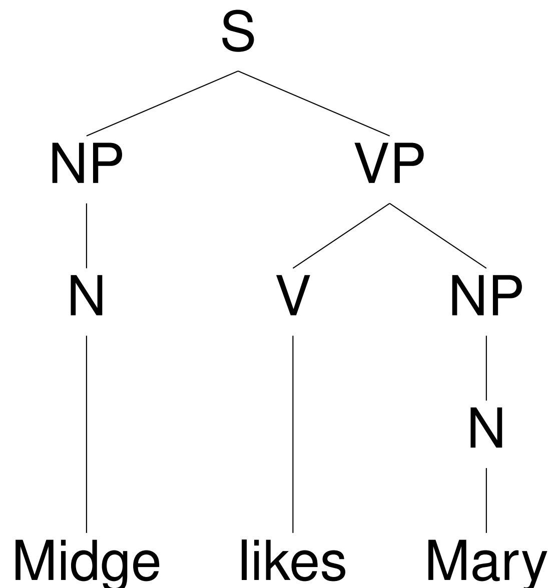
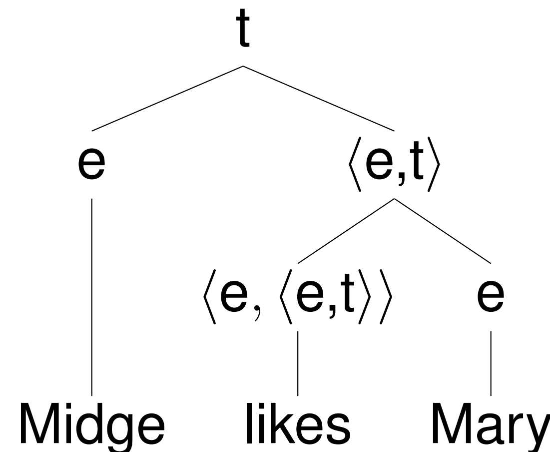
# Semantic Types: Two-Place Predicates

A **transitive verb** requires **two arguments** to be filled in order to form a full sentence, hence it is of the **type** $\langle$**e,**$\langle$**e,t**$\rangle\rangle$.

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# Summary: Types of Expressions

There is a long (potentially infinite) list of **types of expressions** which we might want to represent in our logical language in order to capture the different combinatorial possibilities we find in natural languages.

| **Type** | **Kind of expression** | **Examples** |
|---|---|---|
| $e$ | Individual expression | *John, Jumbo* |
| $\langle e, t\rangle$ | One-place first-order predicate | *walks, red, loves Mary* |
| $\langle e, \langle e, t\rangle\rangle$ | Two-place first-order predicate | *loves, sees* |
| $\langle e, \langle e, \langle e, t\rangle\rangle\rangle$ | Three-place first-order predicate | *lies between (and)* |
| $t$ | Sentence | *John walks, John loves Mary* |
| $\langle t, t\rangle$ | Sentential modifier | *Not* |
| $\langle e, e\rangle$ | Function (entitiy to entity) | *The father of* |
| $\langle\langle e, t\rangle, \langle e, t\rangle\rangle$ | Predicate modifier | *Quickly, beautifully, fast* |
| $\langle\langle e, t\rangle, t\rangle$ | One-place second-order predicate | *Is a color* |
| $\langle\langle e, t\rangle, \langle\langle e, t\rangle, t\rangle\rangle$ | Two-place second-order predicate | *Is a brighter color than* |
| etc. | etc. | etc. |

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# The Syntax: Recursive Definition

The clauses for the syntax of a type-theoretic language are then:

(i) If $\alpha$ is a variable or a constant of type a in *L* [i.e. $v_a$ or $c_a$], then $\alpha$ is an expression of type a in *L*.

(ii) If $\alpha$ is an expression of type $\langle a, b \rangle$ in *L*, and $\beta$ is an expression of type a in *L*, then $(\alpha(\beta))$ is an expression of type b in *L*.

(iii) If $\phi$ and $\psi$ are expressions of type *t* in *L* (i.e. formulas in L), then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.

(iv) If $\phi$ is an expression of type *t* in *L* and v is a variable (of arbitrary type a), then $\forall v\phi$ and $\exists v\phi$ are expression of type *t* in *L*.

(v) If $\alpha$ and $\beta$ are expressions in *L* which belong to the same (arbitrary) type, then $(\alpha = \beta)$ is an expression of type *t* in *L*.

(vi) Every expression *L* is to be constructed by means of (i)-(v) in a finite number of steps.

Gamut (1991), Volume 2, p. 81-82.

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# Examples of Valid and Invalid Expressions

## Definition of Types

Assume j is of type $e$ (i.e. representing an entity), x is of type $e$, A is of type $\langle e, t\rangle$ (i.e. a first order one-place predicate), B is of type $\langle e, \langle e, t\rangle\rangle$ (i.e. a first-order two-place predicate), and $\mathcal{C}$ is of type $\langle\langle e, t\rangle, t\rangle$ (i.e. a second-order one-place predicate).

| Expressions | Clause Applied |
|---|---|
| j ✓ | (i) |
| A ✓ | (i) |
| A(j) ✓ | (i) and (ii) |
| (B(j))(x) ✓ alternative notation: B(j)(x) | (i) and (ii) |
| $\mathcal{C}$(B(j)) ✓ | (i) and (ii) |
| A(j) $\wedge$ $\mathcal{C}$(A) ✓ | (i), (ii), and (iii) |
| $\forall$xA(x) ✓ | (i), (ii), and (iv) |
| Aj x | – |
| B(A) x | – |
| $\forall$x$\mathcal{C}$(x) x | – |

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# Truth Valuation

However, the complexity of defining interpretation functions for all kinds of different types of expressions (see table below from Gamut) is beyond the scope of this course.

Gamut (1991), p. 86.

| **Type** | **Interpretation** |
|---|---|
| $e$ | Entity |
| $\langle e, t\rangle$ | Function from entities to truth values, i.e. characteristic function |
| $\langle e, \langle e, t\rangle\rangle$ | Function from entities to sets of entities |
| $\langle e, \langle e, \langle e, t\rangle\rangle\rangle$ | Function from entities to functions from entities to sets of entities |
| $t$ | Truth value |
| $\langle t, t\rangle$ | Function from truth values to truth values |
| $\langle e, e\rangle$ | Function from entities to entities |
| $\langle\langle e, t\rangle, \langle e, t\rangle\rangle$ | Function from sets of entities to sets of entities |
| $\langle\langle e, t\rangle, t\rangle$ | Characteristic function of a set of sets of entities |
| $\langle\langle e, t\rangle, \langle\langle e, t\rangle, t\rangle\rangle$ | Function from sets of entities to sets of sets of entities |
| etc. | etc. |

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# Section 2: Lambda Calculus in Logical Languages

# Towards a Fully Compositional Account

A logical language $L$ built on the theory of types is extremely powerful, since it is capable of representing a potentially infinite number of different natural language structures. However, we still **cannot (yet) represent parts of sentences** or predicates in a **fully compositional account**.

| **Examples** | **Translations (Typed Language)** |
|---|---|
| John smokes. | $S(j)$ |
| John smokes and drinks. | $S(j) \wedge D(j)$ |
| Every man walks. | $\forall x(M(x) \rightarrow W(x))$ |
| smokes and drinks | ? |
| every man | ? |
| every | ? |

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# Some (Possible?) Translations and Problems

| Example | Translation (?) | Problems |
|---|---|---|
| smokes and drinks | $S \wedge D$ | see (1) |
| smokes and drinks | $S(x) \wedge D(x)$ | see (2) |
| every man | $\forall x M(x)$ | see (3) |
| every | $\forall$, $\forall x$ | see (4) |

(1) This is not a valid expression in a type-theoretic logical language, since only formulas of type *t* can be combined with connectives, while first-order predicates lacking arguments are of type $\langle e, t \rangle$. This translation would not be valid in standard predicate logic either, since atomic sentences are here defined as predicates taking at least one constant or variable.

(2) While this is a valid expression, it would translate as *x smokes and x drinks*, which is not exactly the same as just *smokes and drinks*.

(3) Given a domain of entities *D*, $\forall x M(x)$ backtranslates to *everybody is a man*, which is not the same as *every man*.

(4) Both of these are neither valid expressions in a type-theoretic language, nor valid formulas in standard predicate logic.

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# Historical Note

"[...] it was not until the late 1960s that logical techniques were applied to a **compositional analysis of natural language**, an achievement ascribable mostly to **Richard Montague**. As it turns out, predicate logic is not well suited for this task, one reason being that its expressive power resides exclusively in the **sentence-like category of formulae**: only formulae are recursively combinable, all other expressions are lexical. As long as only the meanings of full sentences are at stake, this does not matter; but when it comes to representing their parts, the **resources of predicate logic** do not suffice."

Zimmerman & Sternefeld (2013), p. 253.

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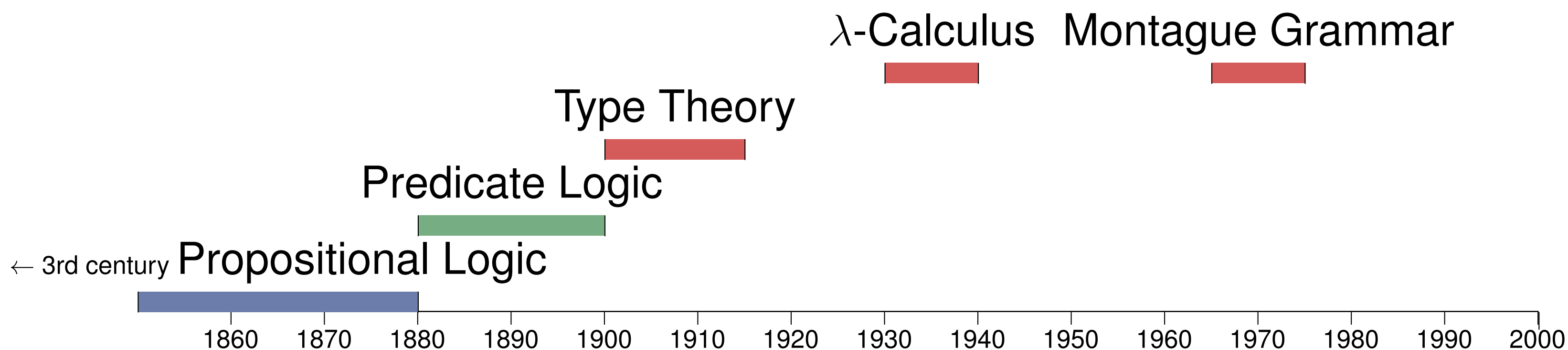
# Historical Note: Lambda Calculus

"**Functional abstraction** was already introduced by Frege (1891), who expressed it by accented Greek letters. Its importance for compositional semantic analysis was first realized by Richard Montague, who used **lambdas** – a notation that goes back to Alonzo Church's (1903–1995) work on the **theory of computation** [...]."

Zimmerman & Sternefeld (2013), p. 254.

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# Disclaimer

“Now it is, of course, also possible to treat the composition of predicates without a $\lambda$-operator [...] **So why do we need a $\lambda$-operator?** The advantage of the $\lambda$-operator is that it provides a **uniform treatment** not only of these examples [combination of predicates] but also of many others too.”

Gamut (1991), Volume 2, p. 107.

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# Section 3: $\lambda$-Abstraction

# The Syntax: Recursive Definition (Last Lecture)

The clauses for the syntax of a **type-theoretic language** are then:

(i) If $\alpha$ is a variable or a constant of type a in *L* [i.e. $v_a$ or $c_a$], then $\alpha$ is an expression of type a in *L*.

(ii) If $\alpha$ is an expression of type $\langle a, b \rangle$ in *L*, and $\beta$ is an expression of type a in *L*, then $(\alpha(\beta))$ is an expression of type b in *L*.

(iii) If $\phi$ and $\psi$ are expressions of type *t* in *L* (i.e. formulas in L), then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.

(iv) If $\phi$ is an expression of type *t* in *L* and v is a variable (of arbitrary type a), then $\forall v\phi$ and $\exists v\phi$ are expression of type *t* in *L*.

(v) If $\alpha$ and $\beta$ are expressions in *L* which belong to the same (arbitrary) type, then $(\alpha = \beta)$ is an expression of type *t* in *L*.

(vi) Every expression *L* is to be constructed by means of (i)-(v) in a finite number of steps.

Gamut (1991), Volume 2, p. 81-82.

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## The Syntax: Adding the $\lambda$-clause

We simply add another clause to the **type-theoretic language** syntax:

(vii) If $\alpha$ is an expression of type a in *L*, and v is a variable of type b, then $\lambda$v($\alpha$) is an expression of type $\langle b, a\rangle$ in *L*.[1]

Gamut (1991), Volume 2, p. 104.

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[1] I added the brackets around $\alpha$ here, since at least in some cases these are necessary to disambiguate.


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# $\lambda$-Abstraction

We say that $\lambda$v($\alpha$) has been formed from $\alpha$ by **abstraction over** the **formerly free variable v**. Hence, the free occurrences of v in $\alpha$ are now **bound by the $\lambda$-operator $\lambda$x**.

Gamut (1991), Volume 2, p. 104.

| **Expression** | **$\lambda$-abstraction** |
|---|---|
| S(x) of type *t* | $\lambda$x(S(x)) of type $\langle e, t \rangle$ |

Note: The first-order predicate S of type $\langle e, t \rangle$ is applied to variable x of type *e*, and yields S(x) of type *t*. In the case of $\lambda$-abstraction this is simply reverted, i.e. the type of $\lambda$x(S(x)) is $\langle e, t \rangle$ again.

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# Examples of $\lambda$-Abstractions

Assume a, b are constants and x, y variables of type $e$; A is of type $\langle e, t \rangle$; B is of type $\langle e, \langle e, t \rangle\rangle$; and X is of type $\langle e, t \rangle$.

| Expressions | Types | $\lambda$-Abstraction | Types |
|---|---|---|---|
| x ✓ | $e$ | $\lambda$x(x) | $\langle e, e \rangle$ |
| A(x)✓ | $t$ | $\lambda$x(A(x)) | $\langle e, t \rangle$ |
| B(y)(x) ✓ | $t$ | $\lambda$x(B(y)(x)) or $\lambda$y(B(y)(x)) | $\langle e, t \rangle$ |
| B(a)(x) ✓ | $t$ | $\lambda$x(B(a)(x)) | $\langle e, t \rangle$ |
| $\forall$xB(x)(y)✓ | $t$ | $\lambda$y($\forall$xB(x)(y)) | $\langle e, t \rangle$ |
| X(a) ✓ | $t$ | $\lambda$X(X(a)) | $\langle\langle e, t \rangle, t \rangle$ |
| X(a) $\wedge$ X(b)✓ | $t$ | $\lambda$X(X(a) $\wedge$ X(b)) | $\langle\langle e, t \rangle, t \rangle$ |

Note: In our practical usage of the type-theoretic language, variables are mostly defined to have type $e$ (i.e. x, y, z, etc.). In some cases, they might be of type $\langle e, t \rangle$, namely, if they refer to predicate variables (X, Y, Z, etc.). Hence, $\lambda$-abstraction essentially amounts to **adding an $e$ or $\langle e, t \rangle$ as a "prefix"** to the type of the expression that is abstracted over.

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# More Examples: Recursive Application

Assume that B below represents the relation of "befriend" in English. The expression (B(y))(x) – here simplified to B(y)(x) – then represents "x befriends y".

| Expressions | Types | $\lambda$-Abstraction | Types |
|---|---|---|---|
| B(y)(x) | $t$ | $\lambda$y(B(y)(x)) | $\langle e, t\rangle$ |
| $\lambda$y(B(y)(x)) | $\langle e, t\rangle$ | $\lambda$x($\lambda$y(B(y)(x))) | $\langle e, \langle e, t\rangle\rangle$ |

**Note:** B(y)(x) is equivalent to the standard predicate logic expression Bxy. We have already pointed out in the last lecture that for two-place predicates it is a convention in type-theoretic languages to apply the predicate first to what would be the object of a transtitive sentence (i.e. y here), and then secondly to what would be the subject (i.e. x). This is also reflected in the order of application of the $\lambda$-abstraction to first y and then x.

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# Section 4: $\lambda$-Conversion

# $\lambda$-Conversion (aka $\beta$-Reduction)

Informally speaking, **$\lambda$-conversion**[2] is the process whereby we reduce the $\lambda$-statement by removing the $\lambda$-operator (and the variable directly following it) and pluging an expression (in the simplest case a constant c, or a predicate constant C) into **every occurrence of the variable which is bound by the $\lambda$-operator**.

| **Typed expression** | **$\lambda$-Abstraction (over x or X)** | **$\lambda$-Conversion (with c or C over x or X)** |
|---|---|---|
| $S(x)$ | $\lambda x(S(x))$ | $\lambda x(S(x))(c) = S(c)$ |
| $S(x) \wedge D(x)$ | $\lambda x(S(x) \wedge D(x))$ | $\lambda x(S(x) \wedge D(x))(c) = S(c) \wedge D(c)$ |
| $X(a) \wedge X(b)$ | $\lambda X(X(a) \wedge X(b))$ | $\lambda X(X(a) \wedge X(b))(C) = C(a) \wedge C(b)$ |

[2] The term $\lambda$-conversion is not to be confused with $\alpha$-conversion. The latter refers to replacing one variable for another.

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# $\lambda$-Conversion (Formal Definition)

In general, **$\lambda$-conversion** is defined as the process whereby an expression of the form $\lambda v(\beta)(\gamma)$ is reduced to $[\gamma/v]\beta$.

- $\beta$ is a typed expression (e.g. S(x) or S(x) $\wedge$ D(x) above).
- v is a variable (e.g. x, y, z, X, Y, Z).
- $\gamma$ is another expression which the $\lambda$-expression is applied to (i.e. functional application). In the simple case, this is a constant c or C as above.
- $[\gamma/v]\beta$ means all occurrences of v in $\beta$ are replaced by $\gamma$.

**Important caveat:** This definition holds only if "all variables which occur as free variables in $\gamma$ are free for v in $\beta$", i.e. if **no occurrence** of v in $\beta$ is bound by a quantifier or another $\lambda$-operator.

Gamut (1991), Volume 2, p. 109-110.

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# Valid and Invalid $\lambda$-Conversions

| $\lambda$-Abstraction | $\lambda$-Conversion |
|---|---|
| $\lambda x(S(x) \wedge D(x))$ | $\lambda x(S(x) \wedge D(x))(c) = S(c) \wedge D(c)$ ✓ |
| $\lambda y(\lambda x(S(x) \wedge D(y)))$ | $\lambda y(\lambda x(S(x) \wedge D(y)))(c)(d) =$ <br> $\lambda x(S(x) \wedge D(c))(d) =$ <br> $S(d) \wedge D(c)$ ✓ |
| $\lambda x(\lambda y(A(y)(x)))$ | $\lambda x(\lambda y(A(y)(x)))(c)(d) =$ <br> $\lambda y(A(y)(c))(d) =$ <br> $A(d)(c)$ ✓ |
| $\lambda x(\forall x F(x))$ | $\lambda x(\forall x F(x))(c)$ **x** |
| $\lambda x(\exists x F(x) \rightarrow S(x))$ | $\lambda x(\exists x F(x) \rightarrow S(x))(c)$ **x** |
| $\lambda X(\forall X(X(a) \wedge X(b)))$ | $\lambda X(\forall X(X(a) \wedge X(b)))(C)$ **x** |

**Note:** While the rule for $\lambda$-abstraction given above in clause (vii) licenses all the abstractions in the left side, $\lambda$-conversion is only valid for a subset of these, namely the ones where the variable v is not bound by a quantifier. In practice, this means we generally avoid $\lambda$-abstractions of variables which are already bound.

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# Section 5: Modelling Compositionality with $\lambda$-Calculus

# Why is $\lambda$-calculus needed?

If our aim is to model not only full sentences and formulas representing predicates, but also parts of sentences, and even individual words, by using in a unified account, then $\lambda$-abstraction and $\lambda$-conversion are possible solutions. Thus, $\lambda$-calculus allows us to capture the **compositionality of language**.

| English sentence | Typed expression |
|---|---|
| *John smokes and drinks.* | $\lambda x(S(x) \wedge D(x))(j) = S(j) \wedge D(j)$ |
| *John smokes* | $\lambda x(S(x))(j) = S(j)$ |
| *smokes* | $\lambda x(S(x))$ |
| *drinks* | $\lambda x(D(x))$ |
| *smokes and drinks* | $\lambda x(S(x) \wedge D(x))$ |

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# Further Examples: Two-Place Predicates

| **English sentence** | **Typed expression** |
|---|---|
| *Jumbo likes Bambi.* | $\lambda x(\lambda y(L(y)(x)))(j)(b) = L(b)(j)$ |
| *Jumbo likes* | $\lambda y(L(y)(j))$ |
| *likes Bambi* | $\lambda x(L(b)(x))$ |
| *likes* | $\lambda x(\lambda y(L(y)(x)))$ |

**Note:** Gamut (1991), Volume 2, p. 107-108, discuss how the active relation "likes" might be represented differently from the passive relation "is liked by". Namely, assume we have L(y)(x) representing "x likes y". If we first abstract over x we get $\lambda x(L(y)(x))$ which represents "likes y" (since x is now bound by the $\lambda$-operator). If we then abstract over y we get $\lambda y(\lambda x(L(y)(x)))$, which represents "likes". If we do it the other way around, however, we first get $\lambda y(L(y)(x))$ which represents "is liked by x", and in the second step we get $\lambda x(\lambda y(L(y)(x)))$ "is liked by". However, we will here assume that active and passive sentences are equivalent in terms of their $\lambda$-representations, otherwise we couldn't represent a structure like "Jumbo likes", since $\lambda y(L(y)(j))$ would strictly translate as "is liked by Jumbo".

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# Further Examples: The Copular "Be"

| **English sentence** | **Typed expression** |
| --- | --- |
| *Jumbo is grey.* | $\lambda$x(G(x))(j) = G(j) |
| *is grey* | $\lambda$x(G(x)) |
| *Jumbo is* | $\lambda$X(X(j)) |
| *is* | $\lambda$X($\lambda$x(X(x))) |

**Note:** In order to just represent be/is here, we have to use a predicate variable X to represent all possible one-place predicates. Since in the last step we abstract both over the variable standing in for the individual (x), and the variable standing in for a predicate applied to the individual (X), all that remains is the copular. Of course, this means that the whole sentence *Jumbo is grey* could also be represented as a $\lambda$-expression, i.e. $\lambda$X($\lambda$x(X(x)))(G)(j) which is equivalent to G(j) after $\lambda$-conversion. This representation of the copular in a $\lambda$-expression is found in Kearns (2011), p. 78.

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# Further Examples: Quantified Expressions

| English sentence | Typed expression |
|---|---|
| *Every man walks.* | $\forall x(M(x) \rightarrow W(x))$ |
| *every man* | $\lambda X(\forall x(M(x) \rightarrow X(x)))$ |
| *every* | $\lambda Y(\lambda X(\forall x(Y(x) \rightarrow X(x))))$ |
| *Some man walks.* | $\exists x(M(x) \wedge W(x))$ |
| *some man* | $\lambda X(\exists x(M(x) \wedge X(x)))$ |
| *some* | $\lambda Y(\lambda X(\exists x(Y(x) \wedge X(x))))$ |
| *No man walks.* | $\neg\exists x(M(x) \wedge W(x))$ |
| *no man* | $\lambda X(\neg\exists x(M(x) \wedge X(x)))$ |
| *no* | $\lambda Y(\lambda X(\neg\exists x(Y(x) \wedge X(x))))$ |

**Note:** In order to represent just the quantifiers *every*, *some*, and *no* we need two predicate variables here (X and Y). Same as before, whole sentences can also be represented by $\lambda$-expressions, e.g. *every man walks* is $\lambda Y(\lambda X(\forall x(Y(x) \rightarrow X(x))))(M)(W)$ before $\lambda$-conversion, which becomes (and is hence equivalent to) $\forall x(M(x) \rightarrow W(x))$ after $\lambda$-conversion.

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# Section 6: The Semantics of $\lambda$-Calculus

# Truth Valuation

As seen before for other logical languages, the semantic side of type-theoretic languages consists of the **valuation of truth** given a syntactically valid expression.

## Example

Remember from last lecture that in a type-theoretic language we might represent a verb like *walks* by W of type $\langle e, t \rangle$. In order to valuate the truth of a particular formula (e.g. W(j)) we define the set of all relevant entities D (the domain) with members d, and a subset $W \subseteq D$ whose members can be said to *walk*. We then define an interpretation function I for which it holds that:

$$I(W)(d) = 1 \text{ iff } d \in W; \text{ and } I(W)(d) = 0 \text{ iff } d \notin W. \quad (1)$$

I(W) is a so-called *characteristic function of W* (over D).

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# Truth Valuation: $\lambda$-Calculus

It is important to realize that **$\lambda$-calculus is not just a syntactic extension to type-theoretic languages**, i.e. some descriptive convention of how to encode certain parts of natural language sentences, but it is also fully compatible with the **semantic side of truth valuation.**

## Example

For an expression W(x) we have the problem that while it is of type *t* we cannot actually assign a truth value $\{0, 1\}$ to it. It can be shown, however, that $\lambda$x(W(x)) is a function h such that

$$h = I(W). \quad (2)$$

In other words, for all entities d in the domain D it holds that h(d)=1 iff I(W)(d)=1. This illustrates that the denotation of $\lambda$x(W(x)) is indeed the same as one would expect for just the word *walks* represented by W.

Gamut (1991), Volume 2, p. 105.

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# Summary

# Summary

- $\lambda$-calculus enables us to **represent the semantic compositionality** of natural language sentences even below the level of predicates and formulas.
- There are two main processes of $\lambda$-calculus, namely, $\lambda$-abstraction and $\lambda$-conversion. The former leads to the binding of formerly unbound variables, the latter can be used to reduce complex $\lambda$-expressions to simpler ones by pluging in the constants.
- The **semantic interpretation** of $\lambda$ expressions is fully compatible with truth valuation as defined for logical languages more generally.

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# Thank You.

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